

4

Electric potential energy



*Fig.4.1.1 © Mike Sharp, Posted to The Cloud Appreciation Society on December 6, 2018.
A storm system over the Bahamas.*

In nature, difference in potential energy between charges in the earth (or water) surface and charges in the lower part of the so-called Cumulo-Nimbus cloud gives the well known discharge effect shown in the figure.

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4.1 Introduction

In this chapter we consider the electric potential energy and in the next chapter we consider the electric potential, which is simply the electric potential energy divided by the charge. The electric potential energy and the electric potential are separated into two chapters, number 4 and 5, in order to avoid confusion of the formulas and because it would become a very large chapter if it became a single chapter.

Charges move through an electric circuit in the direction where their electric potential energy per charge (i.e. the electric potential) decreases. Electric potential energy can be stored in capacitors and it can be lost in resistors.

The understanding of electric potential energy is very important in order to understand electrical circuits with resistors, capacitors, diodes, transistors and batteries as well as many natural phenomena, e.g. a thunderstorm as shown in fig.4.1.1.

4.2 The concept of work and potential energy

We start the chapter by defining the relationship between work performed by a mechanical force and potential energy. Thereafter, the relationship between work performed by an electric force and the electric potential energy is defined.

The basic definition of mechanical work

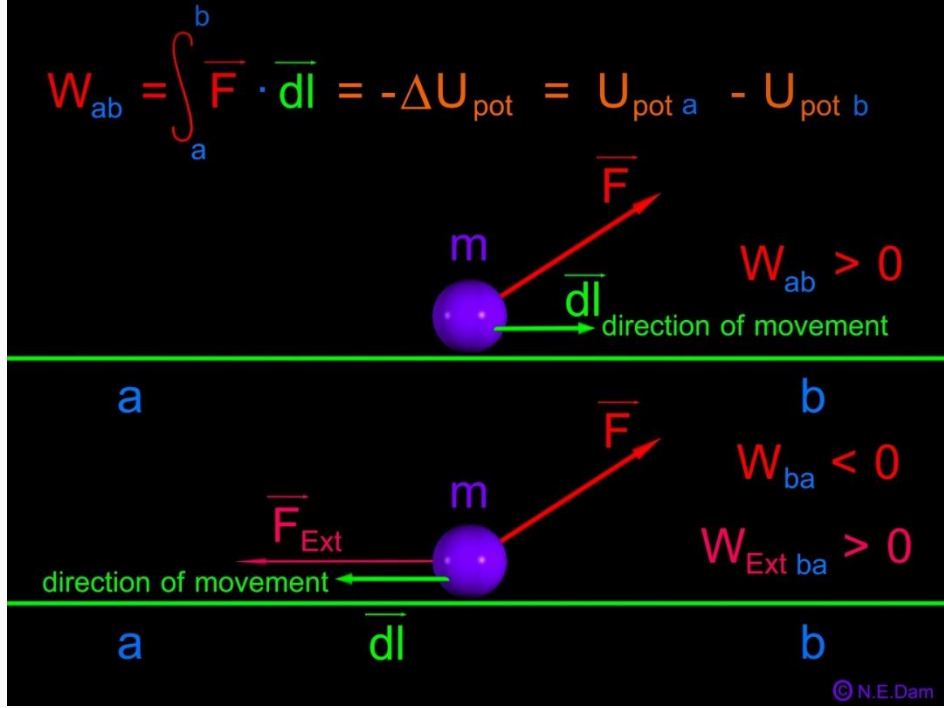


Fig.4.2.1. N.E. Dam ©. The concept of mechanical work. The figure shows that the ball is pulled along a straight line from a to b. The path from a to b can also be a curve, where $\vec{dl}$ points in different tangent directions along this curve. Furthermore, the force can also vary in magnitude and direction along the curve from a to b.

Fig.4.2.1 shows the basic definition of mechanical work. The ball with the mass m is pulled by the force $\vec{F}$ from a to b in the upper part of the figure.

When $\vec{F}$ pulls m the distance $\vec{dl}$ we get the following work performed by the force $\vec{F}$

$$dW_{ab} = \vec{F} \cdot \vec{dl} \quad \text{eq. 4.2.1}$$

In more detail:

$$dW_{ab} = \vec{F} \cdot \vec{dl} = |\vec{F}| |\vec{dl}| \cos \theta = dl \{F \cos \theta\} = dl \{F_p\}, \quad \theta = \angle(\vec{dl}, \vec{F}) \quad \text{eq. 4.2.2}$$

The work is the scalar product $\vec{F} \cdot \vec{dl}$, which is the projection of the force $\vec{F}$ on $\vec{dl}$ times the distance moved dl . The angle from $\vec{dl}$ to $\vec{F}$ is denoted θ . The parenthesis $\{F \cos \theta\}$ is the projection of $\vec{F}$ on $\vec{dl}$ and this projection is denoted F_p . If we replace the force $\vec{F}$ by the projection F_p we get the same work. The unit is force times distance, $N \cdot m = J$. To find the work performed by the force $\vec{F}$ when m is moved from a to b we integrate

$$W_{ab} = \int_a^b \vec{F} \cdot \vec{dl} \quad \text{eq. 4.2.3}$$

According to mechanical physics, the work is equal to the drop in potential energy

$$W_{ab} = -\Delta U_{\text{pot}} = -(U_{\text{pot}, b} - U_{\text{pot}, a}) = U_{\text{pot}, a} - U_{\text{pot}, b} \quad \text{eq.4.2.4}$$

Note, that the increase in potential energy is defined by

$$\Delta U_{\text{pot}} = U_{\text{pot}, b} - U_{\text{pot}, a} \quad \text{eq.4.2.5}$$

If there is another external force $\vec{F}_{\text{Ext}}$ acting on the ball, this force may pull the ball to the left if it is strong enough to overcome the projection of the force $\vec{F}$ on $\vec{dl}$ as shown in the lower part of fig.4.2.1.

The force that “wins” is the force that has the largest projection on the horizontal path. In this case, it is $\vec{F}_{\text{Ext}}$ which has the largest projection. The projection of the total force on the road therefore to the left, i.e. the ball moves to the left from b to a and $\vec{dl}$ points to the left in the direction of movement.

When we calculate the work, $\vec{dl}$ shows the direction of integration, here from b to a. In this case, $\vec{F}_{\text{Ext}}$ performs positive work, as this force is to the same side (even in the same direction) as $\vec{dl}$. At the same time $\vec{F}$ performs negative work as $\vec{F}$ is to the opposite side of $\vec{dl}$.

The sign of the work

The sign of the work is positive if the force $\vec{F}$ is to the same side as the direction of the movement $\vec{dl}$ and negative if the force $\vec{F}$ is to the opposite side of $\vec{dl}$.

The words “to the same side” is not the same as “in the same direction”, see fig.4.2.2 and the explanation below.

The angle θ is the angle from $\vec{dl}$ to $\vec{F}$. This angle is positive in the upper half-plane (i.e. the 1. and 2. quadrant) and negative in the lower half-plane (the 3. and 4. quadrant).

- “In the same direction” means, that $\vec{F}_0$ and $\vec{dl}$ are coparallel, i.e. the angle $\theta = 0$.
- “In the opposite direction” means, that $\vec{F}_6$ and $\vec{dl}$ are antiparallel, $\theta = \pi$ or $\theta = -\pi$, in concentrated form $|\theta| = \pi$.
- “Perpendicular” means, that $\vec{F}_5$ and $\vec{dl}$ or $\vec{F}_7$ and $\vec{dl}$ are orthogonal, i.e. $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$, in concentrated form $|\theta| = \frac{\pi}{2}$.
- “To the same side” means, that $\vec{F}$ is in the right half plane relative to $\vec{dl}$ where the scalar products $\vec{F}_1 \cdot \vec{dl}$ and $\vec{F}_4 \cdot \vec{dl}$ are positive, i.e. $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
- “To the opposite side” means, that $\vec{F}$ is in the left half plane relative to $\vec{dl}$ where the scalar products $\vec{F}_2 \cdot \vec{dl}$ and $\vec{F}_3 \cdot \vec{dl}$ are negative, in concentrated form $\frac{\pi}{2} < |\theta| < \pi$ ¹.

¹ The inequality $\frac{\pi}{2} < |\theta| < \pi$ must be understood in the following way:

If $\theta > 0$ we may remove the absolute sign and we get the interval: $\frac{\pi}{2} < \theta < \pi$ (second quadrant)

If $\theta < 0$ we should replace $|\theta|$ by $-\theta$ when we remove the absolute sign: $\frac{\pi}{2} < -\theta < \pi$. If we multiply by -1 we should reverse the direction of the inequality signs: $-\frac{\pi}{2} > \theta > -\pi$ or $-\pi < \theta < -\frac{\pi}{2}$ (third quadrant).

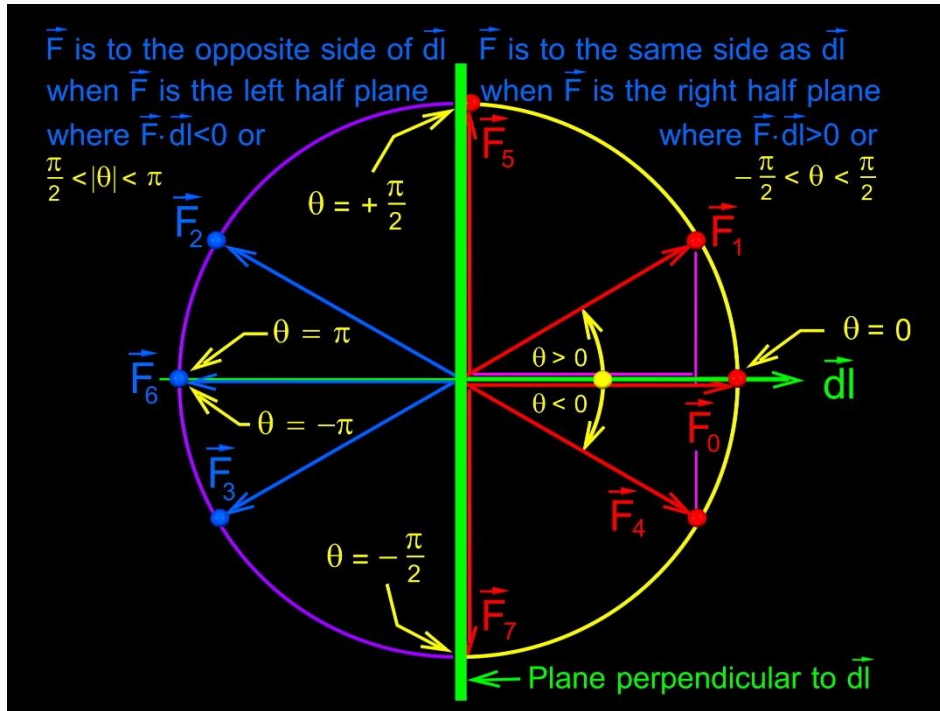


Fig.4.2.2. N.E. Dam ©. Illustration of “to the same side” and to the opposite side”.

The work performed by $\vec{F}$ is positive, if $\vec{F}$ is to the same side as $d\vec{l}$
 The work performed by $\vec{F}$ is negative, if $\vec{F}$ is to the opposite side of $d\vec{l}$
 The work performed by $\vec{F}$ is zero, if $\vec{F}$ is perpendicular to $d\vec{l}$

Work and potential energy when a ball is moving uphill or downhill

We consider the upper part of fig.4.2.3

Vector $d\vec{l}$ is parallel to the surface at the point where the ball is located, i.e. $d\vec{l}$ is parallel to the tangent to the green curve at that point. The gravitational force $\vec{F}_g$ is considered to be a sum of the following two forces:

- $\vec{F}_n$, which is the “normal-force”, is perpendicular to the surface at the point where the ball is located. This force is counteracted by a reaction force $-\vec{F}_n$ in opposite direction.
- $\vec{F}_{gp}$ is the gravitational force projected on the surface.

In addition to the gravitational force $\vec{F}_g$ we apply an external force $\vec{F}_{Ext}$ on the ball. If the projection of the external force on the surface is larger than the projection of the gravitational force on the surface (and in the opposite direction), then the ball will move up the hill with vector $d\vec{l}$ pointing in the direction of movement. At the same time the potential energy of the ball increases and becomes maximum at the top of the hill.

Then we consider the lower part of fig.4.2.3

If we release the ball and remove the external force $\vec{F}_{Ext}$, then the gravitational force $\vec{F}_g$ acts alone and pulls the ball down the hill. The ball is accelerated by the gravitational force on its way downhill and the kinetic energy is increased. At the same

time the potential energy of the ball decreases. The work done by the gravitational force is positive and will be equal to the drop in potential energy.

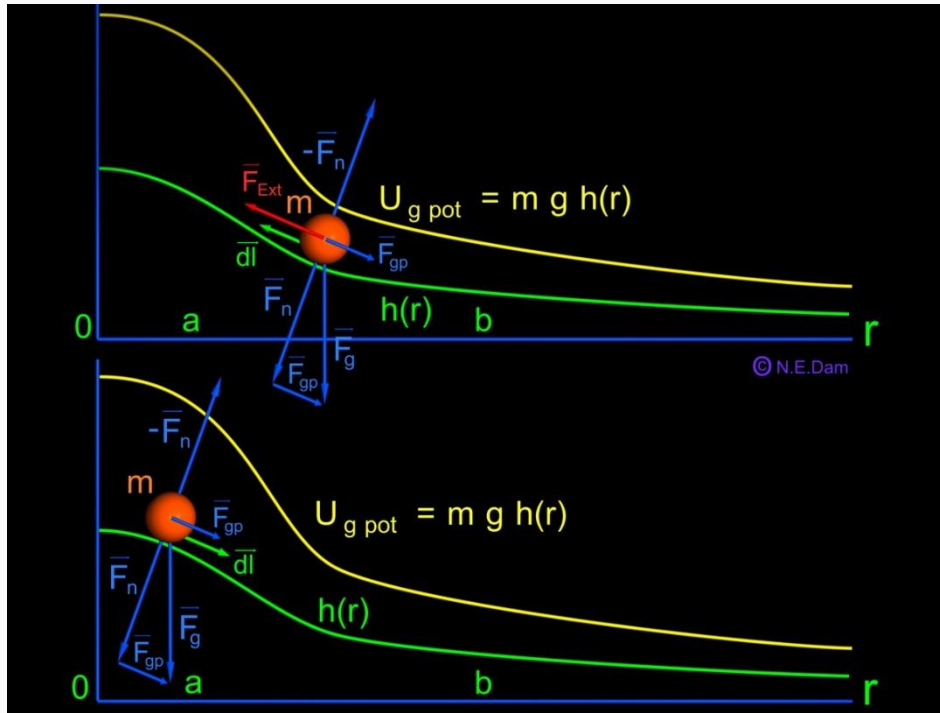


Fig.4.2.3. N.E. Dam ©. Work and potential energy when a ball is pushed uphill.

The concept of a conservative system

Now let us assume, that we have a so-called conservative system without any frictional loss. Such a system “conserves” the total amount of energy. The drop in potential energy will be equal to the increase in kinetic energy:

$$-\Delta U_{\text{pot}} = \Delta U_{\text{kin}} \Rightarrow U_{\text{pot}, a} - U_{\text{pot}, b} = U_{\text{kin}, b} - U_{\text{kin}, a} \Rightarrow$$

$$U_{\text{kin}, a} + U_{\text{pot}, a} = U_{\text{kin}, b} + U_{\text{pot}, b} \quad \text{eq.4.2.6}$$

The sum of kinetic and potential energy is called the mechanical energy and it is constant - or conserved - from a to b.

When a force $\vec{F}$ acts on a particle which moves from a to b, the work done by the force is given by the previously stated curve integral

$$W_{ab} = \int_a^b \vec{F} \cdot d\vec{l} \quad \text{eq. 4.2.3}$$

The work W_{ab} done by a conservative force has the following properties:

1. $W_{ab} = -\Delta U_{\text{pot}}$ (drop in potential energy) $= \Delta U_{\text{kin}}$ (increase in kinetic energy), eq.4.2.6
2. It is reversible $W_{ab} = -W_{ba}$
3. It is independent of the path between a and b
4. It is zero, if the ball is moved from a to b and then back again from b to a.

4.3 Electric potential energy of a charge q_0 in a central E-field from a charge q

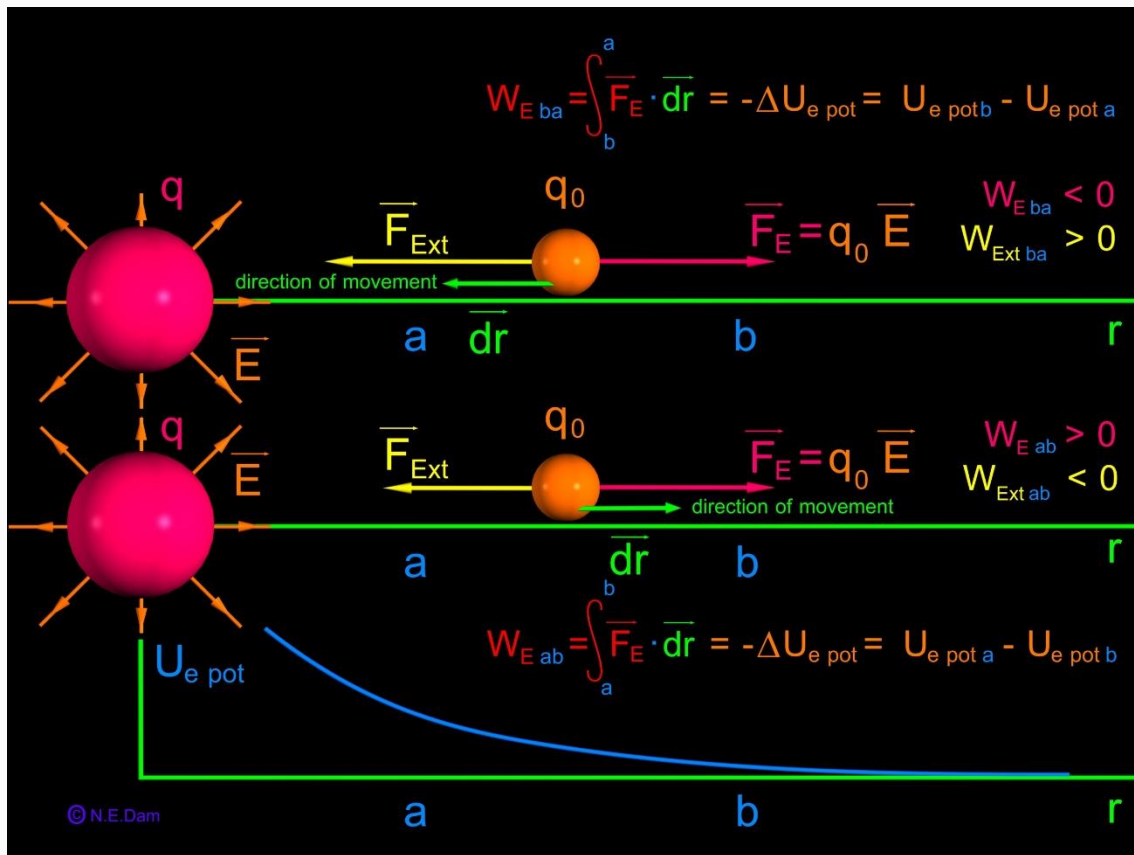


Fig.4.3.1. N.E. Dam ©. Calculation of the electric potential energy of q_0 relative to q . The E-field is generated by q and is spherically symmetrical. The magnitude of the E-field depends only on r . The direction of the E-field is radially away from q . This is a so-called central E-field.

The potential energy of a moving charge

We consider the upper part of fig.4.3.1

We consider a charge q_0 in distance r from another charge q . We imagine, that q stays at the position $r = 0$ while q_0 is moving towards q . If both charges have the same sign (both positive or both negative) then we must apply the force $\vec{F}_{\text{Ext}}$ to q_0 to overcome the electric repulsive force $\vec{F}_E$ and push q_0 towards q . This is analogous to the situation where we push a ball uphill and overcome the gravitational force as shown in the upper part of fig.4.2.3. If we push a positive charge q_0 towards another positive charge q from position b to position a , we (i.e. the force $\vec{F}_{\text{Ext}}$) perform positive work (because $\vec{F}_{\text{Ext}}$ is in the same direction as $d\vec{r}$) when we overcome the repulsion force. When q_0 arrives at position a , q_0 has a higher electric potential energy than at b .

Then we consider the lower part of fig.4.3.1

If we release q_0 at position a , the repulsive electric force does positive work on q_0 when q_0 is accelerated away from q . At the same time the electric potential energy of q_0 decreases.

This is analogous to the situation when the gravitational force pulls a ball downhill as shown in the lower part of fig.4.2.3.

If we leave a ball to itself in a gravitational field, the gravitational force does positive work and pulls the ball downhill while the *gravitational* potential energy decreases. The work done by the gravitational force equals the drop in *gravitational* potential energy. This is analogous to the situation for a charge in an electric field:

If we remove $\vec{F}_{\text{Ext}}$ and leave a positive charge q_0 to itself in an electric field, the electric force will be in the field direction and perform positive work on q_0 while moving q_0 in the direction of the electric field while the electric potential energy decreases. The work done by the electric force equals the drop in electric potential energy.

A negative charge q_0 will also move in the direction of the electric force but this force is in the opposite direction of the E-field. However, q_0 will also move in the direction where the electric potential energy decreases.

We conclude, that q_0 (positive or negative, initially at rest), will move in the same direction as the electric force. It implies, that the electric force performs positive work and the electric potential energy decreases. The direction of movement will be in the same direction as the E-field if q_0 is positive and in the opposite direction if q_0 is negative

If we compare the *electric* potential energy with the *gravitational* potential energy, the *electric* potential energy is typically billions of times larger than *gravitational* potential energy. Hence, for charges and atomic particles, we normally neglect the *gravitational* potential energy and the gravitation force.

Below we will derive an expression for the electric potential energy outlined in fig.4.3.1, where q_0 is in the central E-field generated by q .

Calculation of the electric potential energy

The word “central E-field” means that the E-field is generated by a point charge q placed in the center of the coordinate system. The other charge q_0 is placed somewhere else in the central field generated by q .

The lower part of fig.4.3.1 shows that q_0 is moving radially away from q while the electric potential energy $U_{\text{e pot}}$ is decreasing. The electric force $\vec{F}_E$ between q and q_0 is the Coulomb force. Let us calculate² the work done by the electric force when q_0 is moving from a to b :

$$W_{ab} = \int_a^b \vec{F} \cdot d\vec{l} = \int_a^b \frac{qq_0}{4\pi\epsilon_0 r^2} dr = \frac{qq_0}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right) = U_{\text{e pot, a}} - U_{\text{e pot, b}} \quad \text{eq. 4.3.1}$$

$$\Rightarrow U_{\text{e pot, a}} = \frac{qq_0}{4\pi\epsilon_0} \left(\frac{1}{a} \right) \text{ and } U_{\text{e pot, b}} = \frac{qq_0}{4\pi\epsilon_0} \left(\frac{1}{b} \right) \quad \text{eq. 4.3.2}$$

If the test charge q_0 moves away from the charge q but along a different path, we get the same work since the right side of eq.4.3.1 only depends on the distance between q and q_0 at the starting point (a) and at the end point (b).

² Section 1.2 of Chapter 1 shows how to calculate the integral

$$\int_a^b \frac{1}{r^2} dr = \left[-\frac{1}{r} \right]_a^b = \left[-\frac{1}{b} \right] - \left[-\frac{1}{a} \right] = \frac{1}{a} - \frac{1}{b} \quad \text{eq. 1.2.7}$$

The potential energy is always defined relative to a reference point, where $U_{\text{pot}} = 0$. If we look at eq.4.3.1 and eq.4.3.2, we may choose the reference point to be $b = \infty$ and choose the other point to be the variable point $r = a$. Then we have

$$b = \infty \Rightarrow U_{\text{e pot}, b} = 0 \quad \text{eq. 4.3.3}$$

$$r = a \Rightarrow W_{ab} = U_{\text{e pot}, a} = U_{\text{e pot}} = \frac{qq_0}{4\pi\epsilon_0 r} \quad \text{eq. 4.3.4}$$

If q and q_0 have the same sign, i.e. $qq_0 > 0$, see fig.4.3.2

The electric force on q_0 is to the right and q_0 moves to the right from $a=r$ to $b=\infty$ and radial away from q . We assume, that we have no other forces than the electric force. At $a=r$ we have $U_{\text{e pot}} = \frac{qq_0}{4\pi\epsilon_0 r}$. At $b=\infty$ we have $U_{\text{e pot}} = 0$. Hence, the electric potential energy decreases from $\frac{qq_0}{4\pi\epsilon_0 r}$ to zero. The work done by the electric force when q_0 moves from r to ∞ is positive, because the electric force $\vec{F}_E$ and the displacement $\vec{dr}$ are in the same direction. A positive work means a positive drop (a real drop) in the electric potential energy.

If we were to push q_0 from ∞ to r , the work done by us would also be $\frac{qq_0}{4\pi\epsilon_0 r}$.

This work is also positive, because the external force $\vec{F}_{\text{Ext}}$ exerted by us on q_0 is in the same direction as the displacement $\vec{dr}$. At the same time the electric force performs negative work since the electric force is in the opposite direction of the displacement. The electric potential energy is then increasing.

In conclusion we have:

- $U_{\text{e pot}} = \frac{qq_0}{4\pi\epsilon_0 r}$ is the electric potential energy when q_0 is at the distance r from q .

This equals:

- The work done by the electric force when q_0 moves away from q , from r to ∞ or
- The work done by us if we exert an external force $\vec{F}_{\text{Ext}}$ on q_0 and push q_0 from ∞ to r

If q and q_0 have opposite signs, i.e. $qq_0 < 0$, see fig.4.3.3

Now the electric force of q_0 is to the left as q_0 is attracted by q . If q_0 is moved to the right (by some external force $\vec{F}_{\text{Ext}}$) the electric force performs negative work. This means that we have a negative drop (i.e. an increase) of the electric potential energy when q_0 moves from a to b or from r to ∞ . The electric potential energy increases from a negative value of $U_{\text{e pot}} = \frac{qq_0}{4\pi\epsilon_0 r}$ at distance r to zero at distance ∞ . At the same time we (i.e. the external force $\vec{F}_{\text{Ext}}$) would have to perform positive work when we push q_0 from r to ∞ .

This positive work is called the binding energy of q_0 in the central field from q .

The electric potential energy of q_0 is always defined in relation to at least one other charge q . Section 4.4 describes the electric potential energy of q_0 in relation to several other charges $q_1, q_2, q_3, \dots$

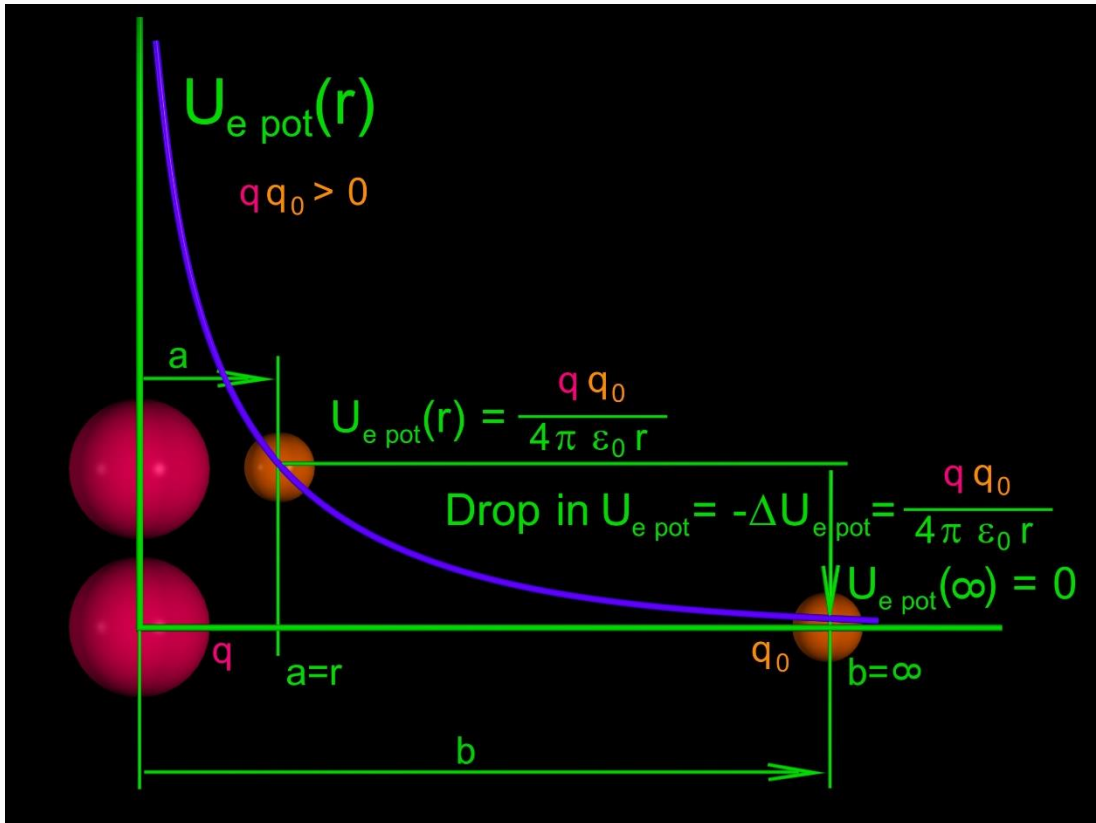


Fig.4.3.2. N.E. Dam ©. The change in $U_{e\text{ pot}}$ when the test charge q_0 is moving away from the larger charge q . The two charges have the same sign, i.e. $qq_0 > 0$.

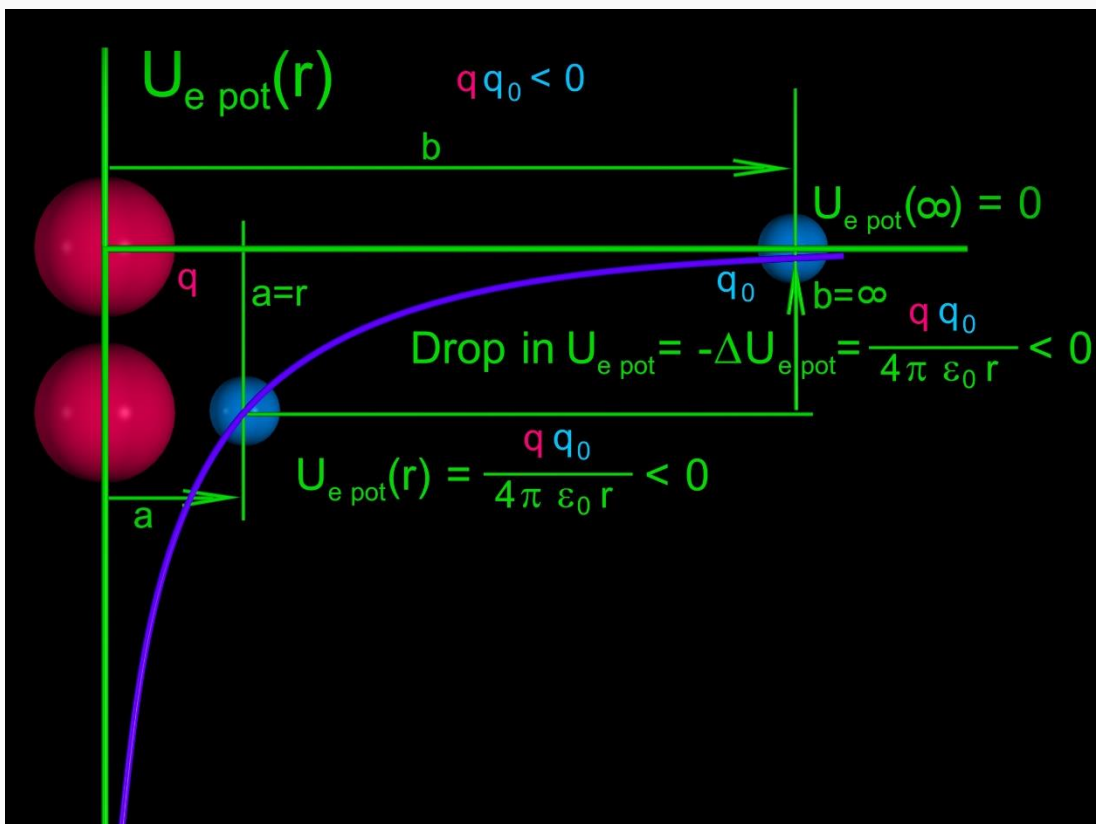


Fig.4.3.3. N.E. Dam ©. The change in $U_{e\text{ pot}}$ when the test charge q_0 is moving away from the larger charge q . The two charges have opposite signs, i.e. $qq_0 < 0$.

Electric potential energy when $q > 0$ and $q_0 < 0$

This is a very important case since it is fundamental to understand the principle of the bond between two oppositely charged ions or between a negative electron and a positive atomic nucleus. According to eq.4.3.4, the electric potential energy of q_0 relative to q is expressed by

$$U_{\text{e pot}} = \frac{qq_0}{4\pi\epsilon_0 r} \quad \text{eq. 4.3.4}$$

The numerator is negative. Hence the curve $U_{\text{e pot}}(r)$ will be the same as in fig.4.3.3. We must invest positive work to push the negative q_0 away from q . Thereby, $U_{\text{e pot}}(r)$ is an increasing function of r .

Now, it is recommended to solve Problem 4.1.

Ionic binding between two atomic ions

Fig.4.3.4 shows a positive and a negative atomic ion. As an example we consider Na^+ and Cl^- . A neutral Na atom has only one electron in its outer shell. Therefore, when it loses this electron and becomes a positive ion, Na^+ , the outer shell will disappear and the atomic radius will be reduced significantly meaning that the radius r_{Na^+} is much smaller than r_{Na} .

A neutral Cl atom has 7 electrons in its outer shell. When Cl gains one more electron and becomes a negative ion, Cl^- , the atomic radius will not change very much, because the outer shell will be the same as before. However, the outer shell becomes filled because the number of electrons reaches 8, which gives a stable outer shell. The 8 electrons (or their wave functions) make a kind of resonance and their binding energy to the nucleus will be relatively high.

The positive Na^+ and the negative Cl^- ion will, of course, attract and approach each other. The electric potential energy will be as shown in fig.4.3.3 or as the blue curve in fig.4.3.5.

Each of the two ions have a negative filled shell of (8) electrons shown as the more intense blue “core” inside the two ions in fig.4.3.4. The two negative “cores” will repel each other. At the same time there will be an attraction between the positive and negative charge at the outer part of the two ions. The electric potential energy due to the repulsive force between the two negative “cores” is shown as the red curve in fig.4.3.5. The repulsion force becomes stronger than the attraction force at short distances. At long distances the attraction force is the stronger of the two forces. The total electric potential energy due to the two forces is the sum of the blue and the red curve in fig.4.3.5. The total electric potential energy, the orange curve, has a minimum where the two opposite directed forces equal each other in magnitude. The center to center distance between the two ions is then $r = r_{\text{eq}}$.

If the distance becomes smaller, $r < r_{\text{eq}}$, the repulsion force will dominate over the attraction force, and opposite if the distance, $r > r_{\text{eq}}$. This keeps the two ions at an equilibrium distance of $r_{\text{eq}} = r_{\text{Na}^+} + r_{\text{Cl}^-}$, which can be measured by using an X-ray crystallographic technique if the material is crystalline.

The binding energy U_{binding} is the energy (the work) that must be performed to remove one ion (e.g. the negative ion) infinitely far away from the positive ion.

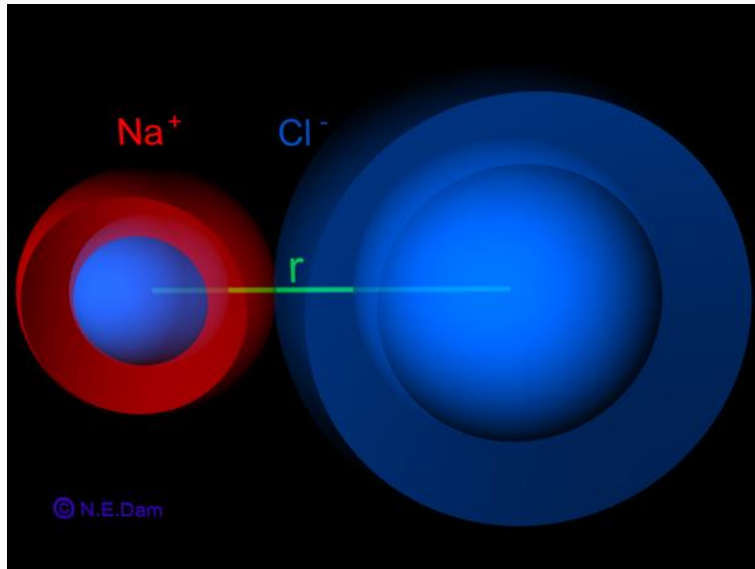


Fig.4.3.4. N.E. Dam ©. Ionic binding between a positive and a negative ion. The ionic radii are as for Na^+ and Cl^- ($r_{\text{Na}^+}=97 \text{ pm}$, $r_{\text{Cl}^-}=181 \text{ pm}$).

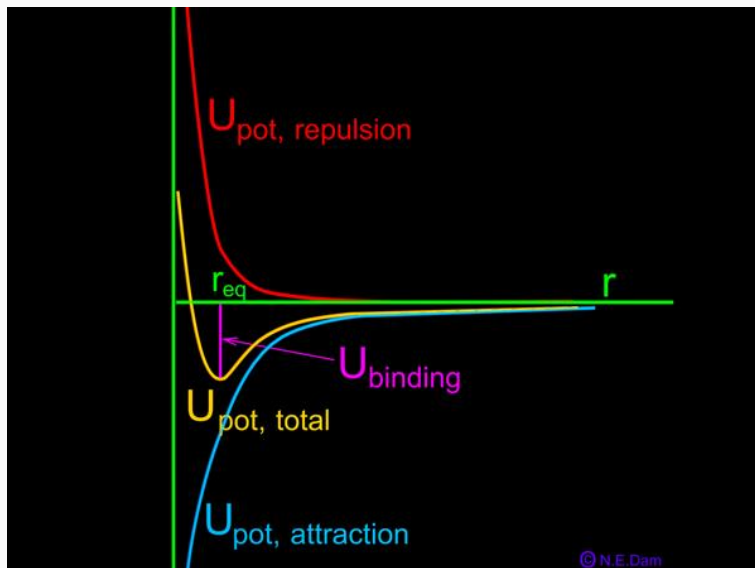


Fig.4.3.5. N.E. Dam ©. Electric potential energy for a system consisting of a positive and a negative ion due to attractive and repulsive forces between the ions. The total electric potential energy has a minimum at an equilibrium distance $r = r_{eq}$.

Binding between an electron and a nucleus

A very basic understanding of the binding of electrons to a nucleus may use the above result for the total electric potential energy. The explanation below is very simple. However, it may give an understanding of binding energy and “binding distance” for electrons orbiting around a nucleus. Each electron has its own curve for the electric potential energy as shown in fig.4.3.6. A more detailed description of this subject is the issue of quantum mechanics or atomic physics. It is a very comprehensive subject and is beyond the scope of this study of electrophysics.

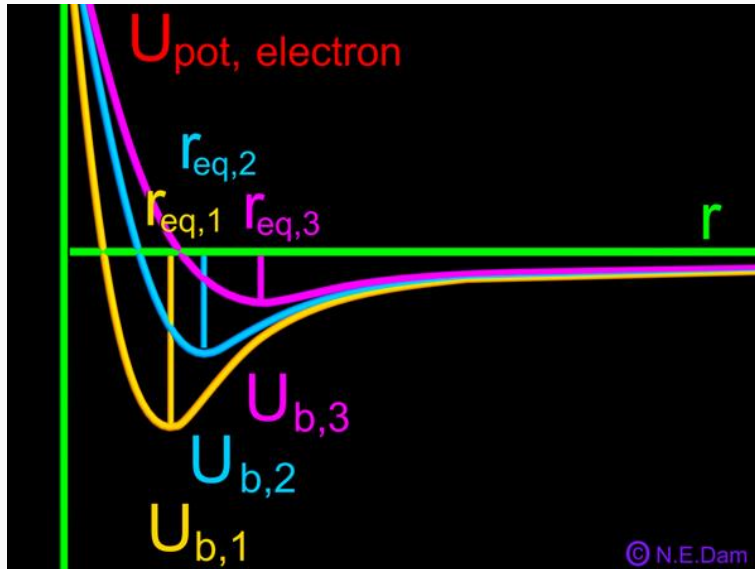


Fig.4.3.6. N.E. Dam ©. Electric potential energy for a negative electron relative to a positive nucleus.

The electrons closest to the nucleus may have an electric potential curve like the orange curve in the figure. The vertical orange line indicates, how much energy or work it requires to excite such an electron an infinite distance away from the nucleus. This is the so-called binding energy $U_{b,1}$ for electron number 1 in the figure. The equilibrium distance between the electron and the nucleus is $r_{eq,1}$. Electrons close to the nucleus (“core electrons”) have a very high binding energy. Electrons at a larger distance from the nucleus (in “higher” shells) have a lower binding energy.

4.4 Electric potential energy of more charges

Electric potential energy when the charges q_1 , q_2 and q_3 are brought together from infinity

We want to calculate the electric potential energy associated with bringing q_1 , q_2 , q_3 together from infinity. Electric potential energy is built up when the charges q_1 , q_2 and q_3 are brought together from infinite distances (on the left side of the blue plane in fig.4.4.1) to finite distances (on the right side of the blue plane).

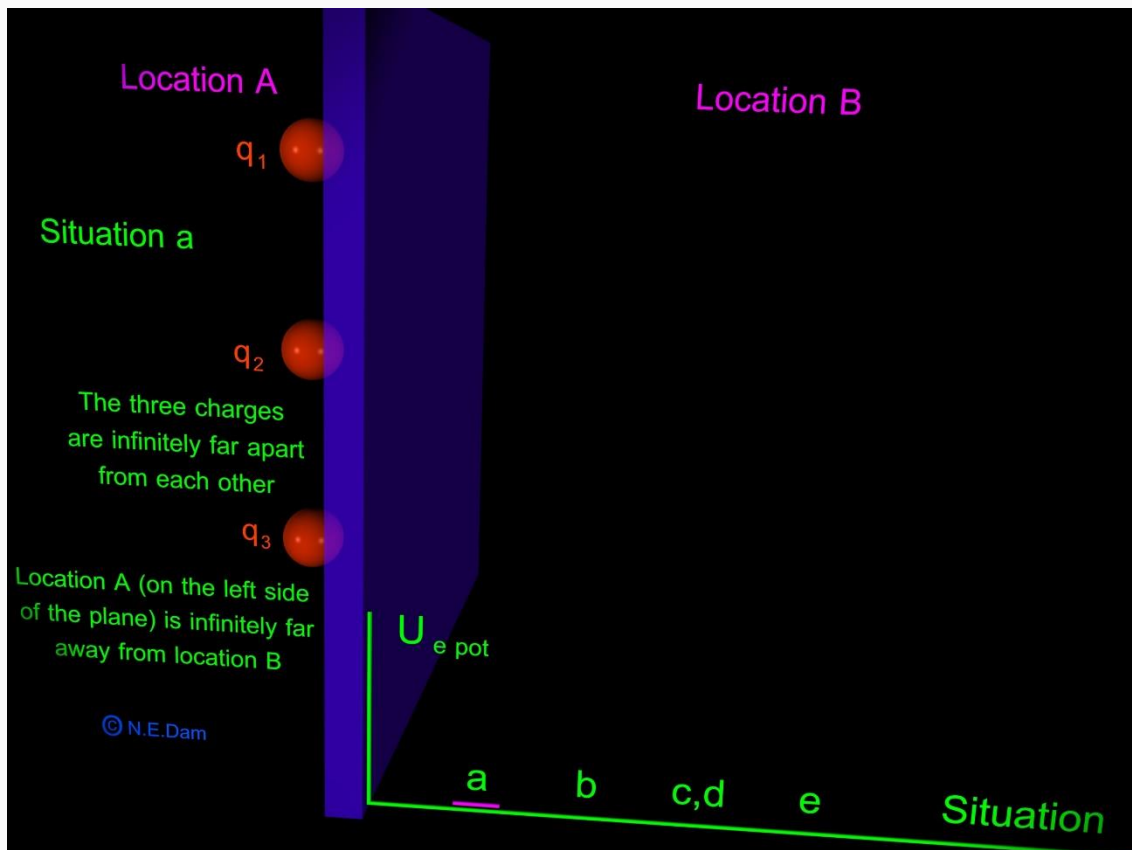


Fig.4.4.1. N.E. Dam ©. Situation a. The three charges q_1 , q_2 and q_3 are on the left side of the blue plane meaning that they are at infinite distances from each other. The electric potential energy is zero.

To begin with, it is easiest to think of all charges as being positive charges. All the involved electric potential energies will then be positive. However, the derived result will also be valid for any combination of positive and negative charges.

Situation a

First we have q_1 , q_2 , q_3 at infinite distances from each other (on the left side of the blue plane in fig.4.4.1). The electric potential energy is zero.

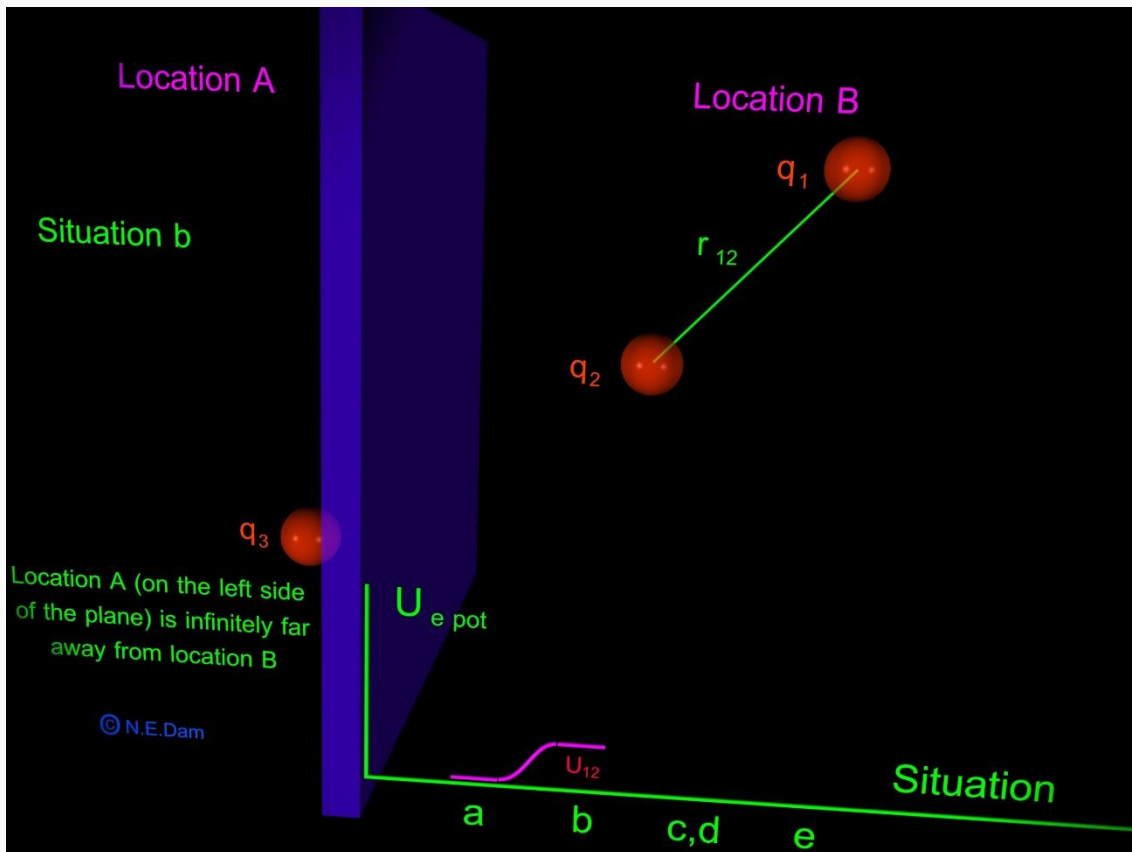


Fig.4.4.2. N.E. Dam ©. Situation b. The two charges q_1 and q_2 are brought together from infinity. The potential energy increases from zero to U_{12} .

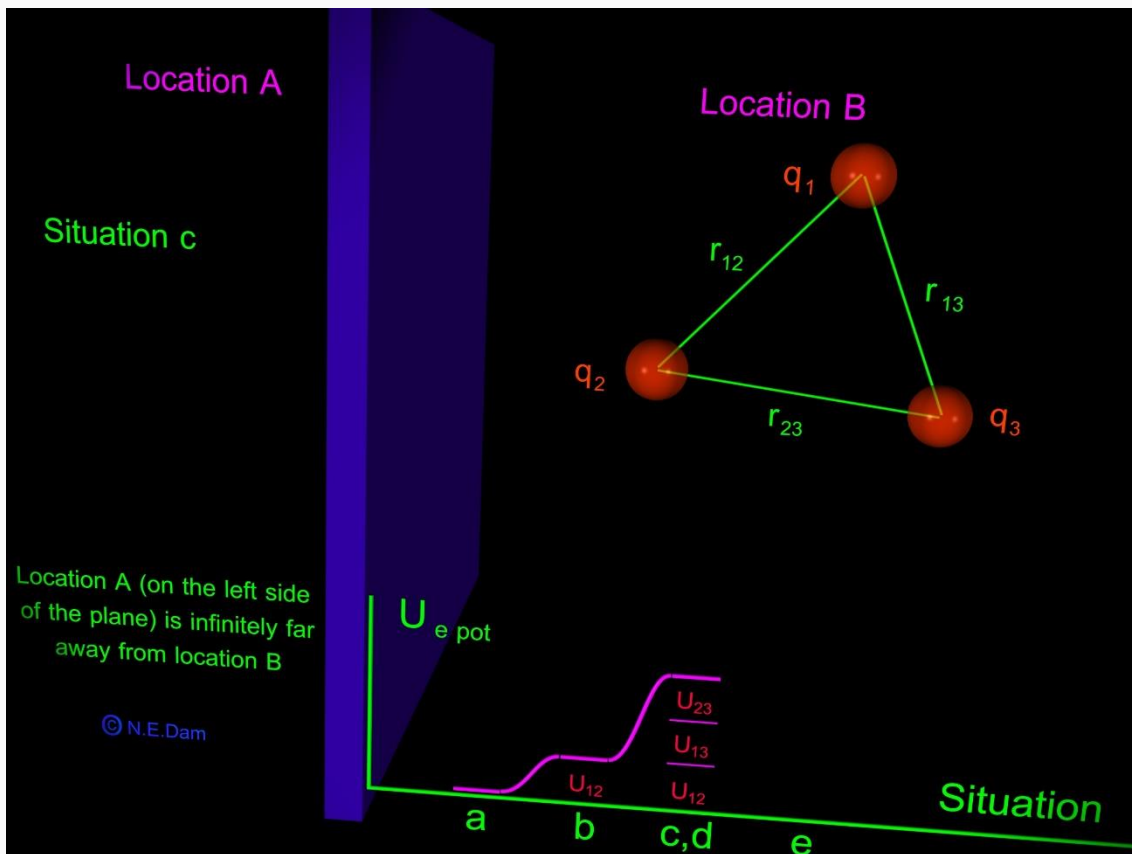


Fig.4.4.3. N.E. Dam ©. Situation c. The third charge q_3 is brought together with q_1 and q_2 . The potential energy increases from U_{12} to $U_{12} + U_{13} + U_{23}$.

Situation b

The two charges q_1 and q_2 are now brought together from infinity. They now appear on the right side of the blue plane in fig.4.4.2 indicating that they are at a finite distance r_{12} from each other. The potential energy increases from zero to U_{12} .

$$U_{12} = U_{\text{e pot}}(q_1, q_2) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_2}{r_{12}} \right\} \quad \text{eq. 4.4.1}$$

It means that the total electric potential energy for the whole system is increased from 0 to this value as shown in fig.4.4.2.

Situation c

Finally, we take q_3 and bring it from infinity to its position on the right side of the blue plane, where q_3 has a finite distance to q_1 and q_2 , see fig.4.4.3. Thereby q_3 acquires electric potential energy relative to q_1 and relative to q_2 . Thereby, the electric potential energy increases by the two terms

$$U_{13} + U_{23} = U_{\text{e pot}}(q_1, q_3) + U_{\text{e pot}}(q_2, q_3) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right\} \quad \text{eq. 4.4.2}$$

Now the total electric potential energy for the whole system is

$$U_{12} + U_{13} + U_{23} = U_{\text{e pot}} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right\} \quad \text{eq. 4.4.3}$$

Note, that we have used the principle of superposition in the above derivation. If we add more charges, we find

$$U_{\text{e pot}} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \dots \right\} = \frac{1}{4\pi\epsilon_0} \sum_{i < j} \frac{q_i q_j}{r_{ij}} \quad \text{eq. 4.4.4}$$

Remarks concerning the summation in the above equation eq.4.4.4

$$U_{\text{e pot}} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \dots \right\} = \frac{1}{4\pi\epsilon_0} \sum_{i < j} \frac{q_i q_j}{r_{ij}} \quad \text{eq. 4.4.4}$$

For simplicity we consider the sum in the case where we have only three charges (with only three terms in the brackets), but the final result is valid for any number of charges.

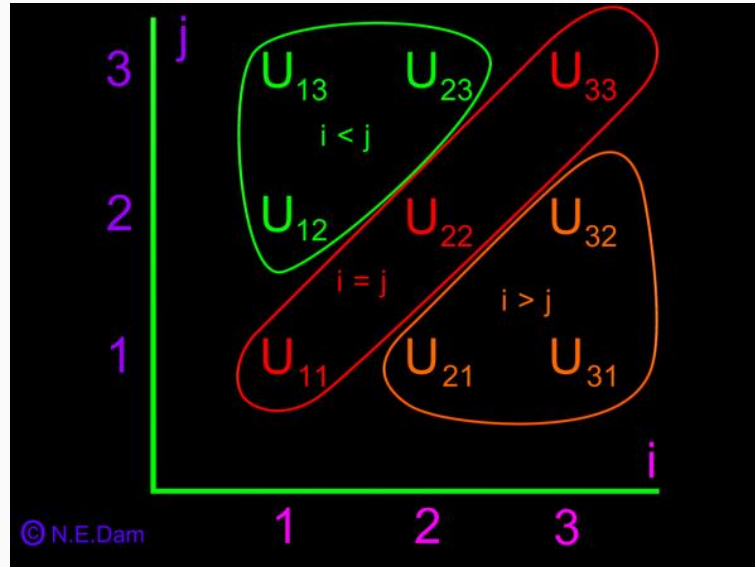


Fig.4.4.4. N.E. Dam ©. Combining i and j in the sum in eq.4.4.4.

If we include all combinations of i and j ($i = 1, 2, 3$ combined with $j = 1, 2, 3$) we get the terms shown in fig.4.4.4. There are too many terms here and we have to exclude the following terms:

- $i = j$ corresponds to the interaction between q_i and q_i . It should not be included in the sum. The electric potential energy is not defined for a charge relative to itself. It should always be relative to other charges.
- $i > j$ gives the same terms as for $i < j$, and we should only include the terms once. Therefore we choose *either* $i > j$ *or* $i < j$ in the sum. In this case, we have chosen $i < j$ in the sum, i.e. only the green terms “survive”.

The electric potential energy when q_0 is moved from infinity and brought together with q_1, q_2, q_3 ..

The following is a continuation of situation a, b and c described above.

Situation d

The three charges q_1, q_2 and q_3 are on the right side of the blue plane in fig.4.4.5 meaning, that they are at finite distances from each other. The electric potential energy of the system is $U_{12} + U_{13} + U_{23}$. An extra charge q_0 is infinitely far away from q_1, q_2 and q_3 , and does not affect the electric potential energy.

We now add an extra charge, the test charge q_0 . As long as q_0 is infinitely far away from q_1, q_2 and q_3 , the presence of q_0 does not influence the electric potential energy. This is situation d shown in fig.4.4.5.

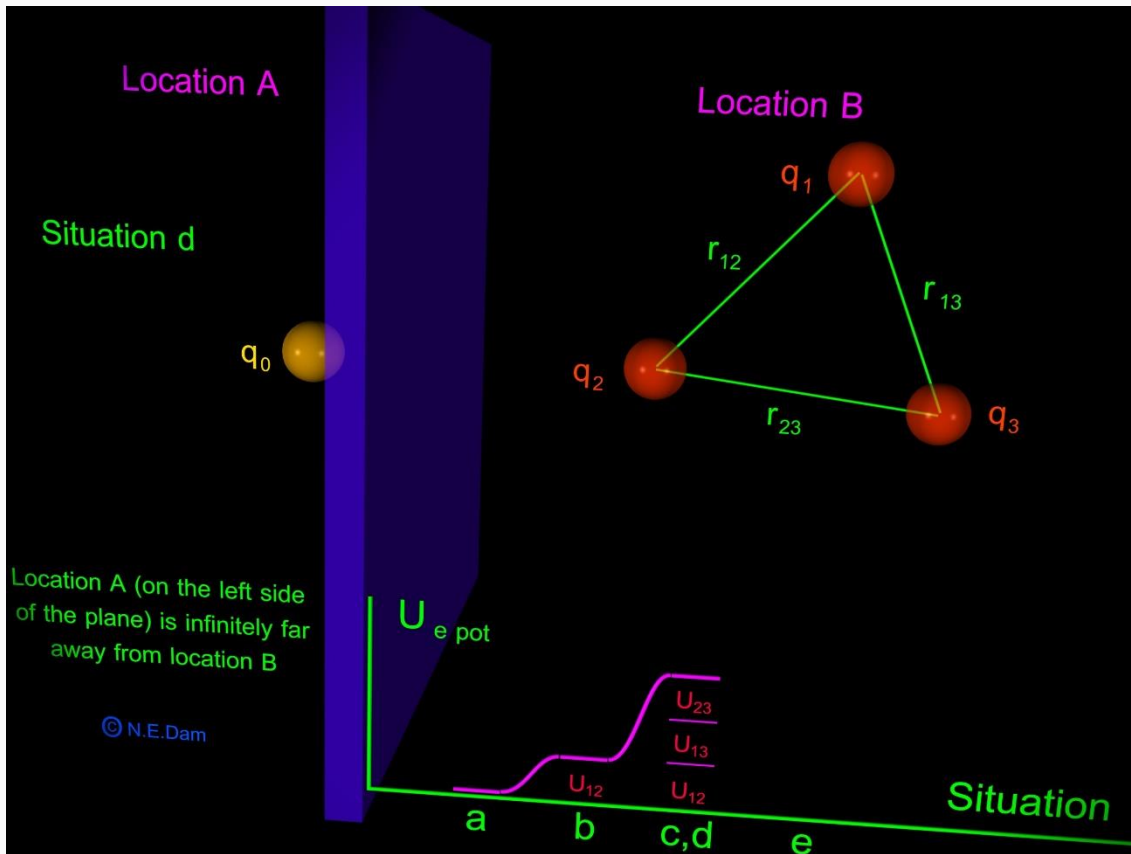


Fig.4.4.5. N.E. Dam ©. Situation d. The three charges q_1, q_2 and q_3 are at finite distances from each other. The charge q_0 , infinitely far away from q_1, q_2, q_3 , does not affect the electric potential energy.

Situation e

The test charge q_0 is now moved towards q_1, q_2, q_3 and placed at some position with finite distances relative to q_1, q_2, q_3 as shown in fig.4.4.6. Thereby, q_0 acquires electric potential energy relative to q_1, q_2 and q_3 . It gives q_0 the electric potential energy

$$U_{e \text{ pot } q_0} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_0 q_1}{r_{01}} + \frac{q_0 q_2}{r_{02}} + \frac{q_0 q_3}{r_{03}} \right\} \quad \text{eq. 4.4.5}$$

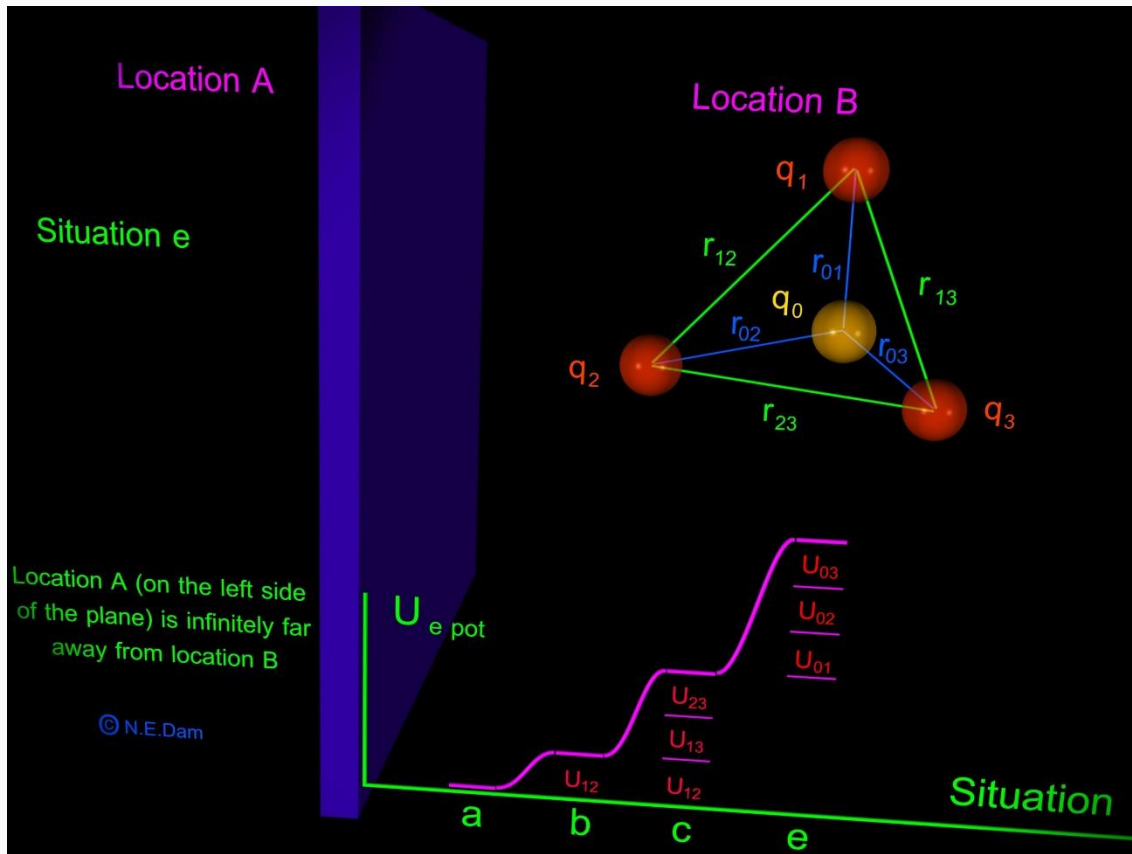


Fig.4.4.6. N.E. Dam ©. Situation e. The charge q_0 is brought together with q_1 , q_2 and q_3 . The electric potential energy thereby increases by $U_{01} + U_{02} + U_{03}$.

This is not the electric potential energy of the entire system, but only the electric potential energy of q_0 relative to q_1 , q_2 , q_3 . In the formula r_{01} is the distance from q_0 to q_1 , likewise for r_{02} and r_{03} . Note, that $U_{e \text{ pot } q_0}$ is zero when q_0 is at an infinite distance from all the charges (the initial state in situation d). If q_0 is brought together with more charges than q_1 , q_2 and q_3 , the formula for the potential energy is expanded this way:

$$U_{e \text{ pot } q_0} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_0 q_1}{r_{01}} + \frac{q_0 q_2}{r_{02}} + \frac{q_0 q_3}{r_{03}} + \dots \right\} = \frac{q_0}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_{0i}} \quad \text{eq. 4.4.6}$$

This is the electric potential energy of q_0 relative to q_1 , q_2 , q_3 , ..., where r_{0i} is the distance from q_0 to q_i .

If we want to calculate the total electric potential energy of the entire system q_0 , q_1 , q_2 , q_3 , we should add the electric potential energy of q_0 relative to q_1 , q_2 , q_3 (eq.4.4.5) to the electric potential energy of q_1 , q_2 , q_3 (eq.4.4.3). However, the total electric potential energy for the entire system can also be calculated using eq.4.4.4 with four charges.

Situation d with negative charges q_1 , q_2 , q_3

This situation should be compared to situation d in fig.4.4.5. The charges q_1 , q_2 , q_3 are positive in fig.4.4.5 and negative in fig.4.4.7. This does not change the potential energies. For example, the product $q_1 q_2$ in U_{12} is positive whether q_1 and q_2 are both positive or both negative.

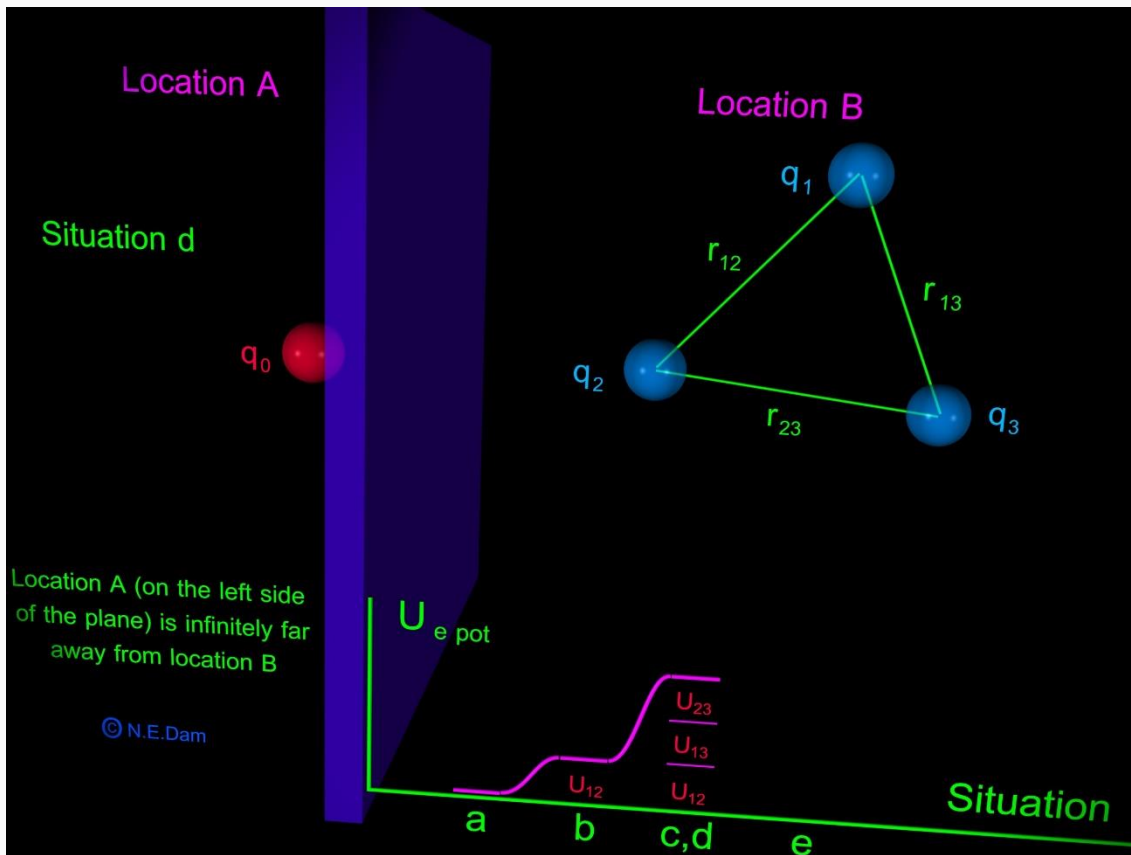


Fig.4.4.7. N.E. Dam ©. Situation d with negative charges q_1, q_2, q_3 . When q_0 is brought together with q_1, q_2, q_3 , the electric potential energy decreases because we add negative values of $U_{01} + U_{02} + U_{03}$.

Situation e with negative charges q_1, q_2, q_3

The positive test charge q_0 is now moved towards q_1, q_2, q_3 and placed at some position with finite distances relative to q_1, q_2, q_3 as shown in fig.4.4.8. Thereby, q_0 acquires negative electric potential energies U_{01}, U_{02}, U_{03} relative to the negative charges q_1, q_2, q_3 .

If q_0 is placed between the charges q_1, q_2, q_3 then the distances from q_0 to q_1, q_2, q_3 will typically be smaller than the mutual distances between q_1, q_2, q_3 . Thereby, the sum of the negative potential energies $U_{01} + U_{02} + U_{03}$ will dominate over the sum of the positive potential energies $U_{12} + U_{13} + U_{23}$. Thereby, the total electrical potential energy becomes negative.

When q_0 is passed from infinitely far away and in between q_1, q_2, q_3 , then the total potential energy decreases³ with $|U_{01} + U_{02} + U_{03}|$ as shown in fig.4.4.8 and in fig.4.4.9.

Conversely, if we are to go backwards and lead q_0 back to infinitely far away from the charges q_1, q_2, q_3 , then the potential energy increases by $|U_{01} + U_{02} + U_{03}|$, ie. we must supply the so-called binding energy $U_{b0} = |U_{01} + U_{02} + U_{03}|$.

Now, it is recommended to solve Problem 4.2.

³ In fact we add the three negative potential energies $U_{01} + U_{02} + U_{03}$ to the three positive potential energies $U_{12} + U_{13} + U_{23}$. Thereby, the total potential energy decreases by the amount $|U_{01} + U_{02} + U_{03}|$.

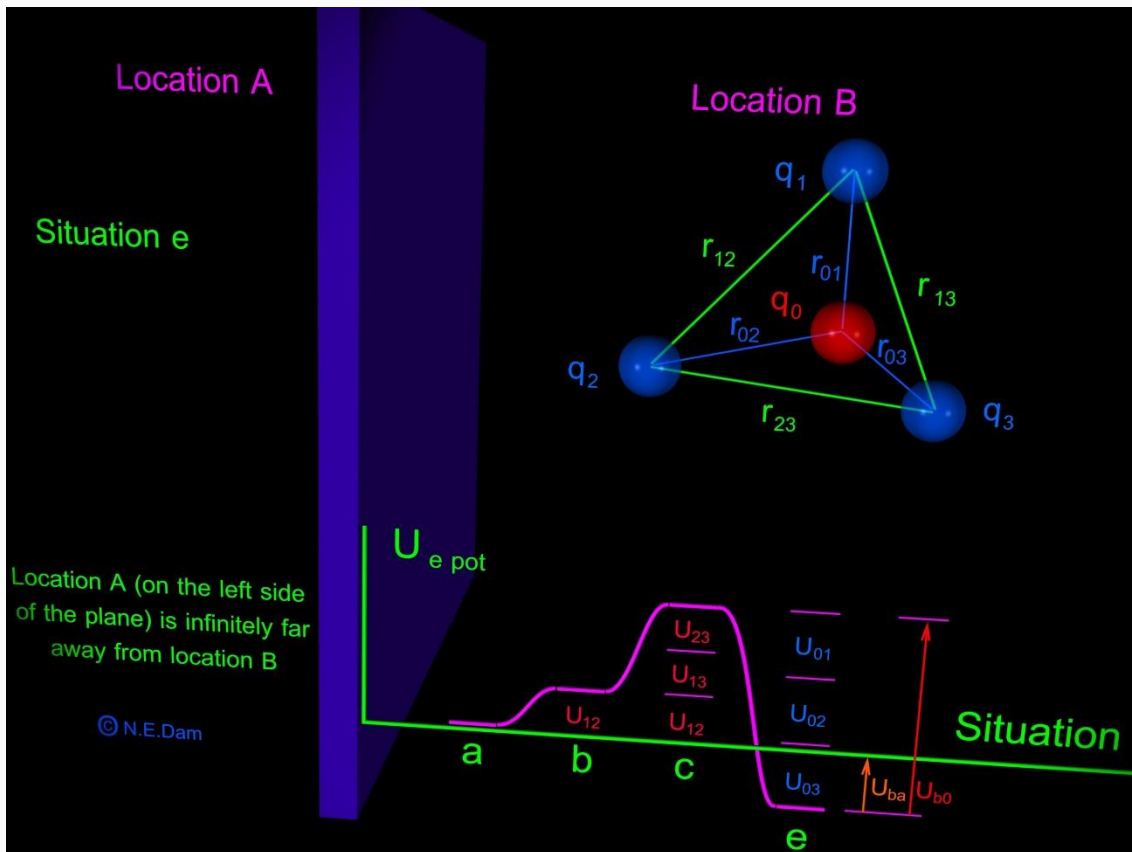


Fig.4.4.8. N.E. Dam ©. Situation e with negative charges q_1 , q_2 , q_3 . When q_0 is brought together with q_1 , q_2 , q_3 , the electric potential energy decreases because we add negative values of $U_{01} + U_{02} + U_{03}$.

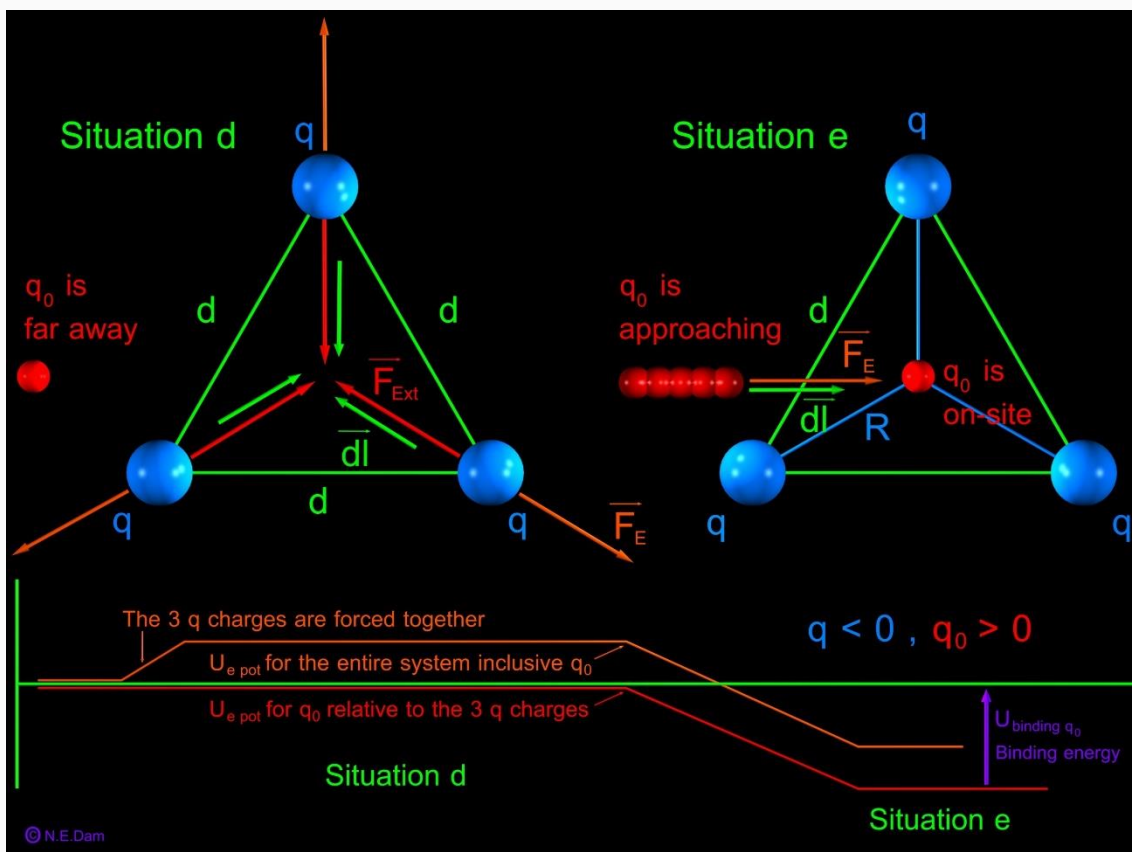


Fig.4.4.9. N.E. Dam ©. The transition from situation d to situation e with negative charges q_1 , q_2 , q_3 .

4.5 Electric potential energy in a uniform field

Fig.4.5.1 shows a positive test charge q_0 , which moves from a to b in a *uniform* E-field between two charged sheets⁴. The force is only in the x direction

$$F = F_x = q_0 E$$

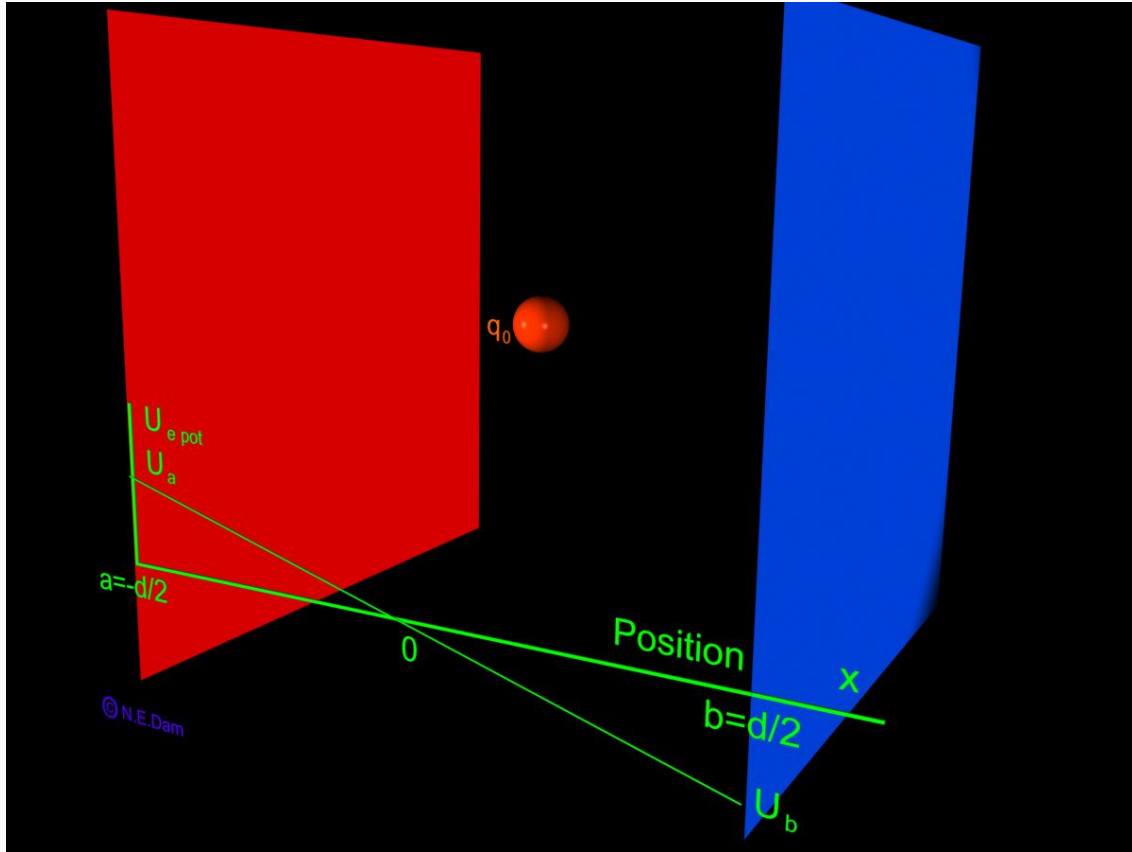


Fig.4.5.1 . N.E. Dam ©. Charge q_0 moving away from the positively charged sheet towards the negatively charged sheet parallel with the x-axis. The E-field in between the two charged sheets is assumed to be uniform. In practice, it requires that the distance between the two sheets is small relative to the extent of the sheets.

The work done by the electric force is

$$W_{ab} = \int_a^b \vec{F} \cdot d\vec{l} = F \int_a^b dx = F_x(b - a) = q_0 E(b - a) = U_a - U_b \quad \text{eq. 4.5.1}$$

Then

$$U_a = -q_0 E a \text{ and } U_b = -q_0 E b \quad \text{eq.4.5.2}$$

⁴ In a plate capacitor we have this situation with a uniform E-field between two charged sheets (the two capacitor plates). The E-field is only uniform if the distance between the two charged sheets (the two capacitor plates) is small relative to the size of the capacitor plates. We return to the capacitor in chapter 6.

The work is positive, since the force has the same direction as the net displacement. We may define this as "the natural direction of movement".

We may define $x = 0$ arbitrarily, and the zero point for the electric potential energy may also be defined arbitrarily. In fig.4.5.1 we have defined $x = 0$ to be in the middle between the two sheets.

If we insert $a = -d/2$ and $b = d/2$ into eq.4.5.2, we find

$$U_a = q_0 E d/2 \text{ and } U_b = -q_0 E d/2 \quad \text{eq.4.5.3}$$

When the test charge moves the distance d from a to b , the total drop in electric potential energy is $q_0 E d$.

If we insert $a = 0$ and $b = x$ into eq.4.5.2, we find $U_a = 0$ and

$$U_x = -q_0 E x \quad \text{eq.4.5.4}$$

which is the electric potential energy of q_0 as a function of x .

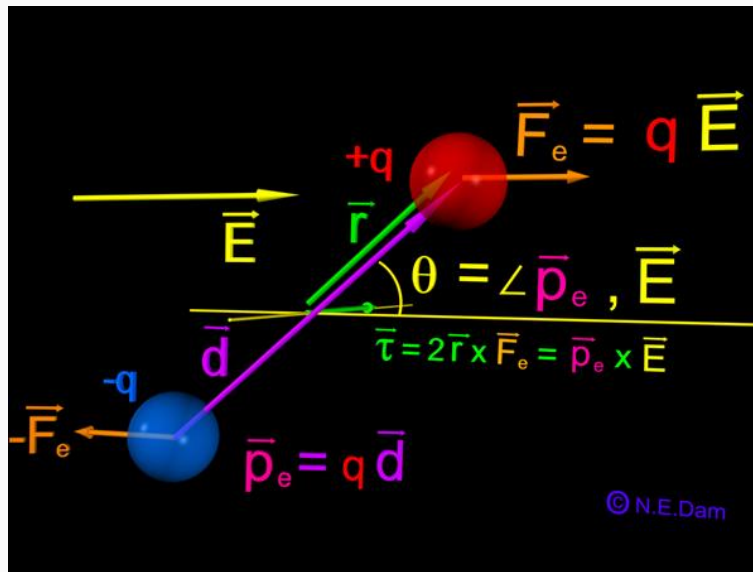


Fig.4.6.1. N.E. Dam ©. Derivation of the torque for an electric dipole in an E-field.

4.6 Electric potential energy of an electric dipole

Repetition of the results from chapter 2

Fig.4.6.1 shows an electric dipole in an E-field. *The electric dipole moment is defined*

$$\vec{p}_e = q \vec{d} = q 2\vec{r} \quad \text{eq. 4.6.1}$$

The torque vector is then

$$\vec{\tau} = 2 \vec{r} \times \vec{F}_e = \vec{p}_e \times \vec{E} \quad \text{eq. 4.6.2}$$

$$\Rightarrow |\vec{\tau}| = |\vec{p}_e| |\vec{E}| \sin \theta \quad \text{eq. 4.6.3}$$

The relation between the work and the torque

As we have seen many times above, the electric potential energy is calculated in the following way:

1. Calculate the work W_{ab} done by the electric force, when a charge is moved from a to b. This work is equal to the drop in electric potential energy $W_{ab} = -\Delta U_{e \text{ pot}} = U_{e \text{ pot a}} - U_{e \text{ pot b}} \quad \text{eq. 4.6.4}$
2. Then decide where the electric potential energy should be equal to zero. E.g. it could be at point b.
3. Finally we have $W_{ab} = U_{e \text{ pot a}} \quad \text{eq. 4.6.5}$

In order to calculate the electric potential energy, we need to calculate the work. Below we see how to calculate the work done by the force when the force is acting on a rotating system like a dipole.

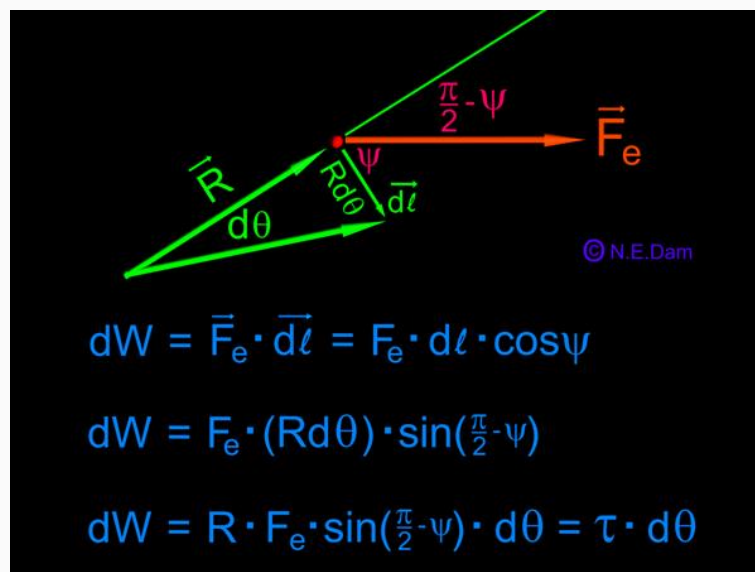


Fig.4.6.2. N.E. Dam ©. The relation between the work and the torque.

Fig.4.6.2 shows the situation when a force is pulling at the red point, which could be the positive charge in a dipole.

As usual, the work is the scalar product between the electric force $\vec{F}_e$ and the displacement $d\vec{l}$

$$dW = \vec{F}_e \cdot d\vec{l} \quad \text{eq. 4.6.6}$$

In words, the work is the projection of the force $\vec{F}_e$ on $d\vec{l}$ times the distance moved dl .

$$dW = \vec{F}_e \cdot d\vec{l} = F_e \cdot dl \cdot \cos(\psi) \quad \text{eq. 4.6.7}$$

where ψ (greek psi) is the angle between $\vec{F}_e$ and $d\vec{l}$.
Since $\cos(\psi) = \sin(\frac{\pi}{2} - \psi)$ and $dl = R d\theta$, we find

$$dW = F_e \cdot (dl) \cdot \cos(\psi) = F_e (R d\theta) \cdot \sin\left(\frac{\pi}{2} - \psi\right) \quad \text{eq. 4.6.8}$$

We have the following for the torque

$$\vec{\tau} = \vec{R} \times \vec{F}_e \Rightarrow |\vec{\tau}| = \tau = |\vec{R}| |\vec{F}_e| \cdot \sin(\angle \vec{R}, \vec{F}_e) = R F_e \cdot \sin\left(\frac{\pi}{2} - \psi\right) \quad \text{eq. 4.6.9}$$

Now eq.4.6.8 becomes

$$dW = \tau d\theta \quad \text{eq. 4.6.10}$$

If the dipole is rotating clockwise in fig.4.6.2, θ is decreasing and $d\theta$ is negative. In order to get positive work, we should use $-d\theta$ instead of $d\theta$ in eq.4.6.10

$$dW = -\tau d\theta \quad \text{eq. 4.6.11}$$

Calculation of the work done by the electric force and the electric potential energy for the electric dipole

Fig.4.6.3 shows the calculation of the work done by the electric force, when the electric force rotates the dipole.

When the dipole rotates from the angle θ_a to θ_b , we use eq.4.6.3 and eq.4.6.11 and get the following calculation of the work done by the electric force

$$\begin{aligned} W_{ab} &= \int_{\theta_a}^{\theta_b} (-\tau) d\theta = \int_{\theta_a}^{\theta_b} (-p_e E \sin \theta) d\theta = [p_e E \cos \theta]_{\theta_a}^{\theta_b} \\ &= p_e E \cos \theta_b - p_e E \cos \theta_a = -\Delta U_{e \text{ pot}} = U_{e \text{ pot } a} - U_{e \text{ pot } b} \quad \text{eq. 4.6.12} \end{aligned}$$

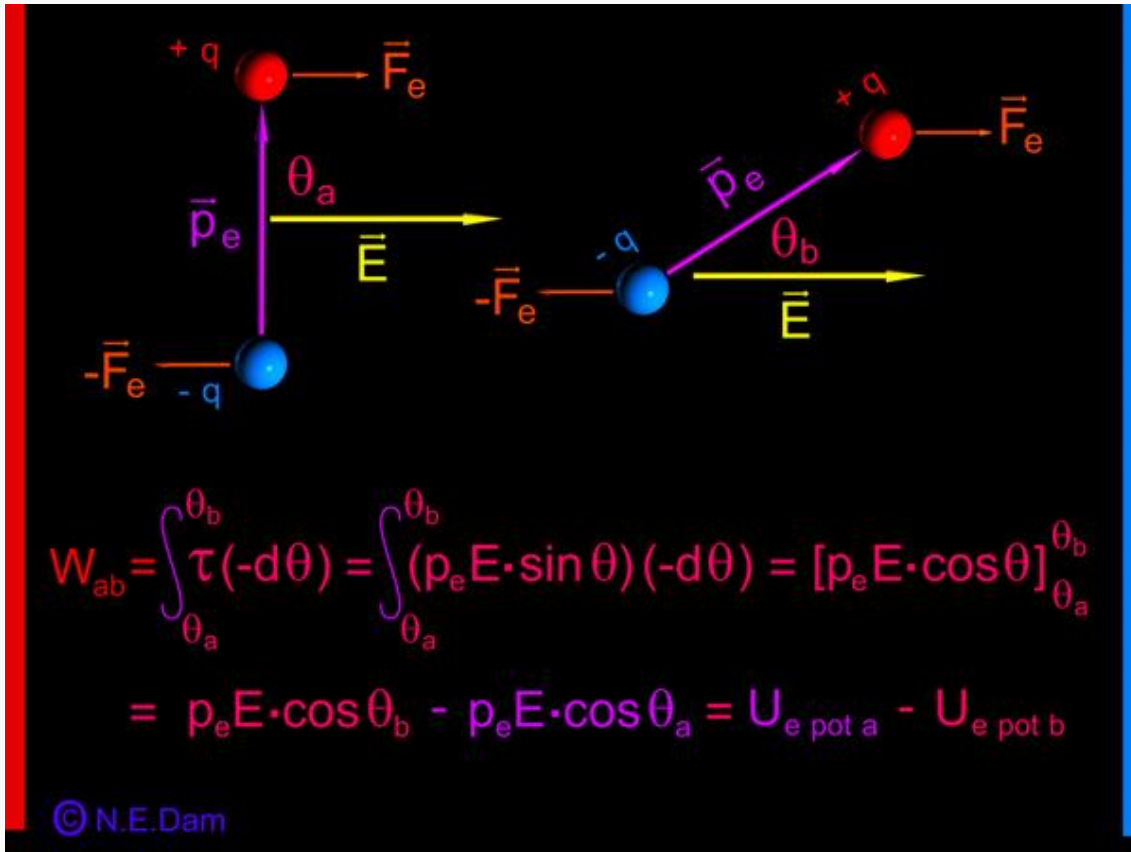


Fig.4.6.3. N.E. Dam ©. Calculation of the work done by the electric force when the electric force rotates the dipole.

From this we identify that

$$U_{e \text{ pot } a} = -p_e E \cos \theta_a \quad \text{eq. 4.6.13}$$

and

$$U_{e \text{ pot } b} = -p_e E \cos \theta_b \quad \text{eq. 4.6.14}$$

If we want the zero point for the electric potential energy at

$$\theta_a = 90^\circ \quad \text{eq. 4.6.15}$$

we get

$$U_{e \text{ pot } a} = -p_e E \cos \theta_a = 0 \quad \text{eq. 4.6.16}$$

If we then remove index b in eq.4.6.14, we get the general expression for the electric potential energy

$$U_{e \text{ pot}} = -p_e E \cos \theta = -\vec{p}_e \cdot \vec{E} \quad \text{eq. 4.6.17}$$

According to eq.4.6.3 we have for the torque

$$\tau = p_e E \sin \theta \quad \text{eq. 4.6.3}$$

Fig.4.6.4 shows the torque and the electric potential energy for an electric dipole using the two equations above (eq.4.6.3 and eq.4.6.17). The figure also shows the position of the dipole relative to the electric field.

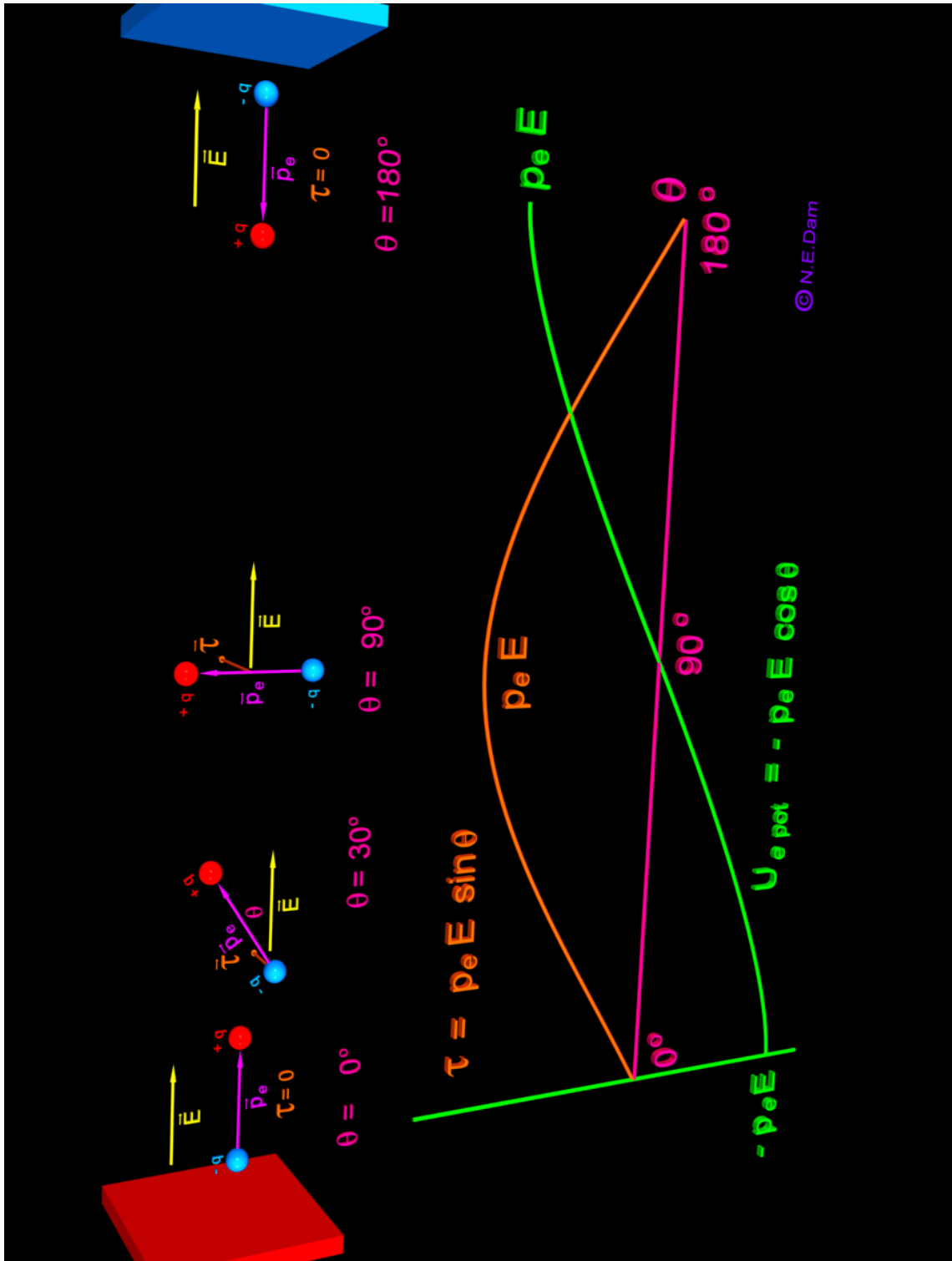


Fig.4.6.4. N.E. Dam ©. The torque and the electric potential energy for an electric dipole.